ON NUCLEAR SCATTERING

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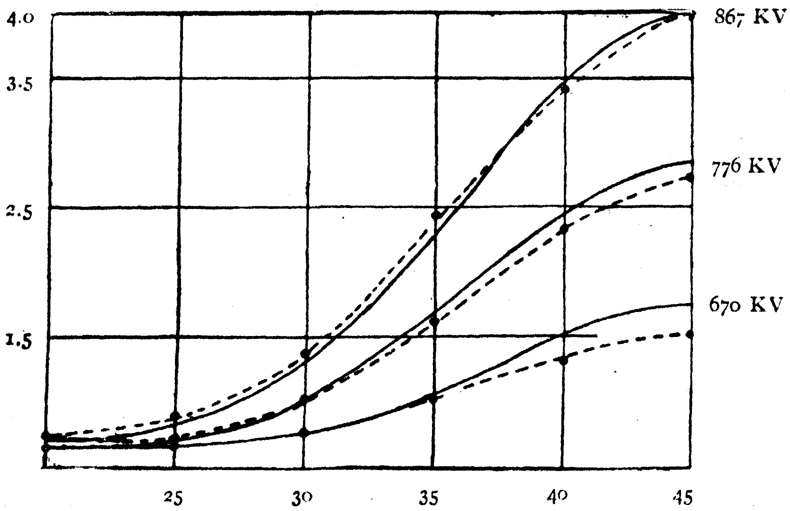
ABSTRACT. The formula for the intensity of proton-proton scattering given by Kar and Basu (*Phil Mag.*, Vol. **29**, p. 200, 1940) is shown to be in better agreement with the second and improved experiment of Heydenburg, Tuve and Hafstad (*Phys. Rev.*, Vol. **56**, p. 1078, 1939). The formula was derived assuming the existence of a short range attractive force of neutron-proton type in addition to the Coulombian force of repulsion. The corresponding formula for the deuteron-deuteron scattering is deduced in the present paper assuming a short range repulsive force of the same type; and it is found to be in good agreement with the observations of Heydenburg and Roberts (*Phys. Rev*, Vol. **56**, p. 1092, 1939). Again, assuming a short range attractive force and neglecting the effect of exchange, a formula is derived for the intensity of scattering of deuteron by helium. The agreement with the experiment of Heydenburg and Roberts is fairly satisfactory. It is also shown that the correction for the relative motion of the scattering particle is not negligible in case of deuteron scattering by helium but may be neglected even at large angles in the case of proton scattering by helium. It is in agreement with the above experiment.

A. Proton-Proton Scattering

Recently the theory of proton-proton scattering has been developed by me.[1] The theory is based on the assumption that in addition to the Coulomb repulsive force there is a short range attractive force between the protons. The latter force is of the nature of the attractive force existing between a neutron and a proton. The formula thus derived for the departure (I/I_{MKB}) from Coulomb scattering, is

$$\frac{I}{I_{MKB}}=\frac{\operatorname{cosec}^4\theta+\sec^4\theta-\operatorname{cosec}^2\theta\sec^2\theta\left(1+\frac{Mv^2}{e^2}g\right)+\left(\frac{Mv^2}{e^2}g\right)^2}{\operatorname{cosec}^4\theta+\sec^4\theta-\operatorname{cosec}^2\theta\sec^2\theta}, \quad \cdots \quad (1)$$

where M, e, v have their usual meaning and g denotes the relative value of the scattering function due to the short range force, being taken independent of the scattering angle (θ) and the incident velocity (v) as in neutron-proton scattering.[2] It should be noted that g is independent of the angle and the velocity only for high-velocity bombardments. Consequently formula (1) is strictly valid in this region and is only approximately true at low velocity. Now, in the paper cited above[1] the theoretical values given by (1) have been compared with the experimental values of Heydenburg, Tuve and Hafstad.[3] The agreement has been fairly good for 900 k.v., 800 k.v. and 700 k.v. incident velocities. Recently they have repeated the experiment[4] more carefully and have found slight discrepancies. The incident velocities in their second experiment are 867 k.v., 776 k.v. and 670 k.v.



Angle of Scattering
Proton-Proton Scattering

FIGURE 1

In Fig. 1, the values of I/I_{MKB} obtained from the second experiment of Heydenburg and others, for different angles of scattering are given by the dotted curves while the corresponding theoretical curves from (1) are drawn continuous. The agreement appears to be better than before. The fitting value of g at $45°$ for 867 k.v., which is used for theoretical calculations, is 4.949×10^{-13}.

B. Deuteron-Deuteron Scattering

The short range force in case of deuteron-deuteron interaction may be taken as before spherically symmetrical and independent of the incident velocity at least to a first approximation, although on account of the complexity in structure the field should have axial symmetry. Moreover, we have to suppose that in the present case, unlike the proton-proton and proton-neutron field, the short range force is *repulsive*. This appears to be supported by the observations of Heydenburg and Roberts who find that the ratio (I/I_{MKB}) of the observed value of the intensity to that given by the MKB-formula is greater than unity even at small angles.

Now remembering that the short range force is repulsive and the weights for anti-parallel and parallel spins are as 2:1, one obtains for deuteron-deuteron scattering, on proceeding exactly as in deriving (1),

$$\frac{\mathrm{I}}{\mathrm{I}_{\mathrm{MKB}}}=\frac{\operatorname{cosec}^4\theta+\sec^4\theta+\frac{2}{3}\operatorname{cosec}^2\theta\sec^2\theta\left(1+4\frac{\mathrm{M}v^2}{e^2}g\right)+\frac{8}{3}\left(\frac{\mathrm{M}v^2}{e^2}g\right)^2}{\operatorname{cosec}^4\theta+\sec^4\theta+\frac{2}{3}\operatorname{cosec}^2\theta\sec^2\theta},$$

where it should be noted, g is taken to be spherically symmetrical as before. The experimental values of the ratio for different angles and for incident velocities 832 k.v., 720 k.v. and 614 k.v. are given by Heydenburg and Roberts. Their experimental curves are drawn dotted in Fig. 2. The corresponding theoretical curves from (2) are drawn continuous.

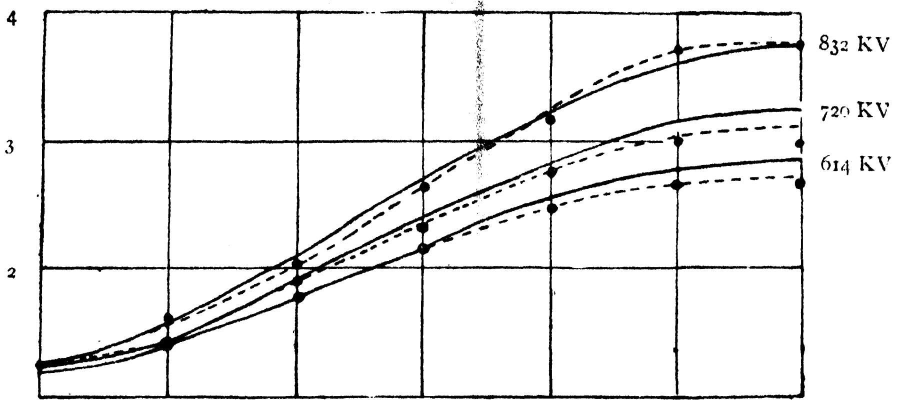


FIGURE 2

It appears from Fig. 2 that the agreement is fairly satisfactory. A better agreement, specially for low velocities, would have been desirable. The fitting value of g at 45° and for 832 k.v. incident velocity is found to be 1.605×10^{-13}.

C. *Deuteron-Helium Scattering*

Heydenburg and Roberts (*l.c.*) have also determined the intensity of scattering of deuteron by helium, and have given its values for different angles relative to the corresponding theoretical values calculated from Born-Rutherford formula. They have taken this ratio probably because they have thought that as there is no possibility of exchange, Born-Rutherford formula is applicable in this case if the force of interaction is Coulombian.

It is clear that as the scattered deuteron and the recoil helium are easily distinguishable, the effect of exchange need not be considered. However, one should not use Born-Rutherford formula because it does not take into account (1) the idea of critical approach found necessary in explaining electron scattering by atoms [6] and (2) the error due to neglecting the velocity of the scattering particle. In the following it is proposed to derive general formulae of scattering and critical approach for any two masses of the interacting particles in a Coulomb field. This enables us at once to estimate the order of the error involved in taking Born-Rutherford formula for deuteron-helium scattering.

Let us suppose that a particle of mass m_2 is at rest and that it is bombarded by a second particle of mass m_1 moving with velocity v. If we refer the motion to the centre of gravity of the combined system (C-system), it can be easily shown that the total energy of the particles in this system is $E=\frac{1}{2}\mu v^2$, where $\mu=\frac{m_1 m_2}{m_1+m_2}$. Thus we have for the wave equations in the C-system, without and within the potential field,

$$\Delta x+\frac{8\pi^2\mu}{h^2}Ex=0 \text{ (outside)}$$

$$\Delta x+\frac{8\pi^2\mu}{h^2}(E-V)x=0 \text{ (inside)},$$

where $$V=\frac{zz'e^2}{r},$$

ze and $z'e$ being the charges of the two interacting particles.

Now, Eqs. (3) and (4) are almost identical with the corresponding equations in electron scattering. The important difference is that here we have μ the effective mass in place of m the electronic mass of the previous case. Thus proceeding in the same way as before we obtain for the relative intensity of scattering at an angle ϕ

$$I=2\pi\left(\frac{zz'e^2}{2\mu v^2}\right)^2 \operatorname{cosec}^4\frac{\phi}{2}\cos^2 k'r_0 \sin\phi\, d\phi, \quad \dots \quad (5)$$

where $$k'=\frac{4\pi\mu v}{h}\sin\frac{\phi}{2} \quad \dots \quad (5\cdot1)$$

and $$r_0=1\cdot35\times\frac{zz'e^2}{\mu v^2}\left(\operatorname{cosec}\frac{\phi}{2}+1\right). \quad \dots \quad (5\cdot2)$$

It should be noted that ϕ in eqs. (5)→(5.2) denotes the angle of scattering in the C-system. Thus in order to verify the above equation experimentally, it is necessary to correlate ϕ with θ the angle of scattering in the laboratory co-ordinates (R-system). We shall proceed to do it in the following :—

As already taken, m_1 is the mass of the incident particle bombarding with velocity v and m_2 the mass of the scattering particle being initially at rest. Thus the total energy in the R-system is $\frac{1}{2}m_1v^2$. Let v_1 and v_2 be the velocities of the two particles in C-system. Elementary considerations give

$$\left.\begin{aligned} v_1 &= \frac{m_2}{m_1+m_2}v \\ v_2 &= -\frac{m_1}{m_1+m_2}v \end{aligned}\right\} \qquad \dots \quad (6)$$

as the velocities, and $\frac{1}{2}\mu v^2$ as the total energy in the C-System. Now, as $\frac{1}{2}m_1v^2 - \frac{1}{2}\mu v^2$ is equal to the energy due to the motion of the centre of gravity of the system, we have, for the velocity of $c.g.$,

$$\frac{m_1}{m_1+m_2}v \qquad \dots \quad (6\cdot1)$$

Again, in the C-system, the scattered particle has components of velocity

$$v_1 \cos\phi = \frac{m_2}{m_1+m_2} v \cos\phi \qquad \dots \quad (6\cdot2)$$

parallel to the line of centres and

$$v_1 \sin\phi = \frac{m_2}{m_1+m_2} v \sin\phi \qquad \dots \quad (6\cdot3)$$

perpendicular to it. In R-system the perpendicular component is unaffected but from (6·1) the parallel component becomes

$$\frac{m_1+m_2\cos\phi}{m_1+m_2}v. \qquad \dots \quad (6\cdot4)$$

Thus taking the ratio of the components in R-system we obtain at once

$$\frac{m_2 \sin\phi}{m_1+m_2\cos\phi} = \tan\theta. \qquad \dots \quad (6\cdot5)$$

where θ is the angle of scattering in R-system. After easy transformations we get from (6·5)

$$\phi = \theta + \sin^{-1}\left(\frac{m_1}{m}\sin\theta\right). \qquad \dots \quad (6\cdot6)$$

Hence,
$$\frac{\sin\phi\, d\phi}{\sin\theta\, d\theta} = 2\frac{m_1}{m_2}\cos\theta + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta}}. \qquad \dots \quad (6\cdot7)$$

On using the relations (6·6) and (6·7) we get for the relative intensity scattering at angle θ, in laboratory co-ordinates,

$$I=2\pi\left(\frac{zz'e^2}{2\mu v^2}\right)^2 \operatorname{cosec}^4\left\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}\left(\frac{m_1}{m_2}\sin\theta\right)\right\}\cos^2 k'r_0$$

$$\times\left\{2\frac{m_1}{m_2}\cos\theta+\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta}}\right\}, \quad \dots \quad (7)$$

where

$$k'=\frac{4\pi\mu v}{h}\sin\left\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}\left(\frac{m_1}{m_2}\sin\theta\right)\right\}, \quad \dots \quad (7\cdot1)$$

and for the critical approach

$$r_0=1\cdot35\times\frac{zz'e^2}{\mu v^2}\left[\operatorname{cosec}\left\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}\left(\frac{m_1}{m_2}\sin\theta\right)\right\}+1\right]. \quad \dots \quad (7\cdot2)$$

Formulæ (7)⟶(7·2) are valid for interaction between *any two masses* under Coulomb field. However, some special cases of importance are worth considering.

Case I: If $m_2 >> m_1$, Eqs. (7)⟶(7·2) become

$$I=2\pi\left(\frac{zz'e^2}{2m_1v^2}\right)^2\operatorname{cosec}^4\frac{\theta}{2}\cos^2 k'r_0\sin\theta\, d\theta. \quad \dots \quad (8)$$

$$k'=\frac{4\pi m_1 v}{h}\sin\theta \quad \dots \quad (8\cdot1)$$

$$r_0=1\cdot35\times\frac{zz'e^2}{m_1v^2}\left[\operatorname{cosec}\frac{\theta}{2}+1\right]. \quad \dots \quad (8\cdot2)$$

For bombardment by very high speed light particles $r_0\longrightarrow 0$ and $\cos k'r_0\longrightarrow 1$. Hence (8) reduces to the well-known Rutherford formula.

Case II: If $m_1=m_2=m$, $z'=z=1$; so $\mu=\frac{m}{2}$. Hence eqs. (7)⟶(7·2) become

$$I=2\pi\left(\frac{e^2}{mv^2}\right)^2\operatorname{cosec}^4\theta\cos^2 k'r_0\cdot 4\cos\theta\sin\theta\, d\theta \quad \dots \quad (9)$$

where

$$k'=\frac{2\pi mv}{h}\sin\theta \quad \dots \quad (9\cdot1)$$

and

$$r_0=1\cdot35\times\frac{2e^2}{mv_2}(\operatorname{cosec}\theta+1). \quad \dots \quad (9\cdot2)$$

Eq. (2) gives the relative intensity for proton-proton scattering without exchange. So it corresponds to an ideal case.

Case III : In case of scattering of deuteron by helium, we have $m_2=2m_1$ and $z=1$, $z'=2$. And so $\mu=\frac{2}{3}m_1$. Hence we have from eqs. (7)⟶(7·2)

$$I=2\pi\cdot\frac{9}{4}\left(\frac{e^2}{m_1v^2}\right)^2\operatorname{cosec}^4\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}(\tfrac{1}{2}\sin\theta)\}\cos^2k'r_0$$

$$\times\left\{\cos\theta+\frac{1+\frac{1}{4}\cos 2\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}\right\}\sin\theta\,d\theta \quad \ldots \quad (10)$$

$$k'=\frac{\frac{8}{3}\pi m_1v}{h}\sin\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}(\tfrac{1}{2}\sin\theta)\} \quad \ldots \quad (10\cdot1)$$

$$r_0=1\cdot35\times\frac{3e^2}{m_1v^2}\left[\operatorname{cosec}\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}(\tfrac{1}{2}\sin\theta)\}+1\right], \quad \ldots \quad (10\cdot2)$$

where m_1 is the mass of deuteron.

Case IV : In case of scattering of helium by deuteron, we have $m_2=\frac{1}{2}m_1$ and $z=2$, $z'=1$. And so $\mu=\frac{1}{3}m_1$. Hence we obtain from eqs. (7)⟶(7·2)

$$I=2\pi\times9\left(\frac{e^2}{m_1v^2}\right)^2\operatorname{cosec}^4\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}(2\sin\theta)\}\cos^2k'r_0$$

$$\times\left\{4\cos\theta+\frac{1+4\cos 2\theta}{\sqrt{1-4\sin^2\theta}}\right\}\sin\theta\,d\theta \quad \ldots \quad (11)$$

$$k'=\frac{\frac{4}{3}\pi m_1v}{h}\sin\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}(2\sin\theta)\} \quad \ldots \quad (11\cdot1)$$

$$r_0=1\cdot35\times\frac{6e^2}{m_1v^2}\left[\operatorname{cosec}\{\tfrac{1}{2}\theta+\tfrac{1}{2}\sin^{-1}(2\sin\theta)\}+1\right] \quad \ldots \quad (11\cdot2)$$

where m_1 is the mass of helium. It is evident that in eqs. (11)⟶(11·2), $2\sin\theta$ must be less than 1. Thus $\sin\theta<<1$, *i.e.*, $\theta<<30°$. Hence the maximum angle at which helium may be scattered by deuteron is 30°. It may be noted that similar conclusion was arrived at by Rutherford and Darwin.

Let us return once again to deuteron scattering by helium. It is now clear that instead of taking after Heydenburg and Roberts (*l.c.*), the intensity relative to that given by Born-Rutherford formula one should rather take the ratio with the intensity given by (10). The error made in taking Born-Rutherford formula is obviously [*vide* (8) and (10)]

$$D_1\,.\,D_2 \quad \ldots \quad (12)$$

where D_1 and D_2, the corrections for relative motion and critical approach respectively, are given by

$$D_1=\frac{4}{9}\cdot\frac{\operatorname{cosec}^4\frac{1}{2}\theta}{\operatorname{cosec}^4\{\frac{1}{2}\theta+\frac{1}{2}\sin^{-1}(\frac{1}{2}\sin\theta)\}}\cdot\frac{1}{\cos\theta+\dfrac{1+\frac{1}{4}\cos 2\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}} \quad \ldots \quad (12\cdot1)$$

and $$D_2 = \frac{1}{\cos^2 k' r_0} \quad \ldots \quad (12{\cdot}2)$$

in which k' and r_0 are given in (10·1) and (10·2). In columns (4) and (5) of Table I we give the values of D_1 and corrected I/I_0 for different angles of scattering. It should be noted that D_1 is independent of the incident velocity. The experimental values of Heydenburg and Roberts (*l.c.*) are given in column 3. It is evident that the correction is quite appreciable at large angles. The second correction D_2 for the critical approach varies from 5.95×10^2 at 20° to 2.29 at 75° for incident velocity 880 k.v. Thus it follows that at 20° Rutherford's formula gives a value (as $D_1 \sim 1$) 5.95×10^2 times greater than that given by (10).

TABLE I

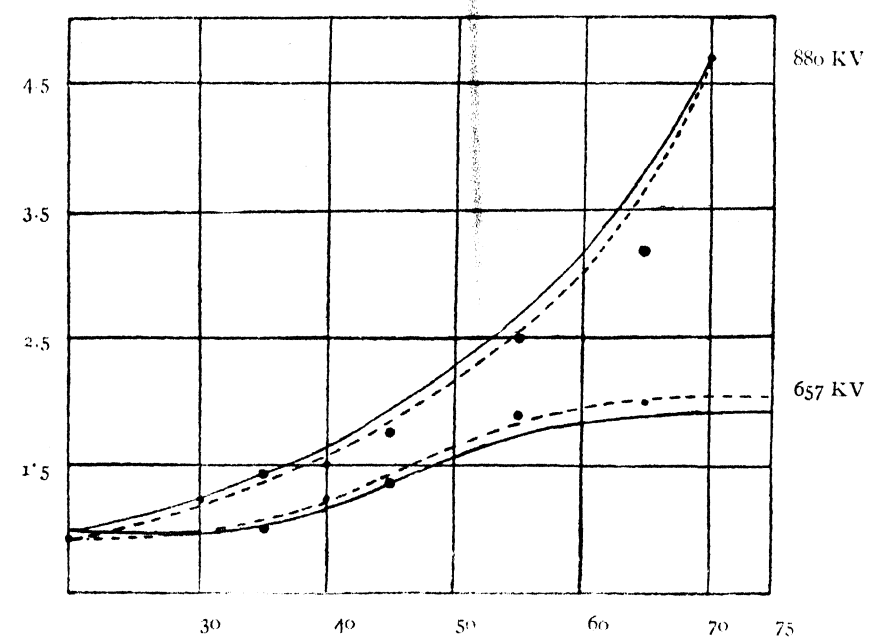
Incident Volt	Angles of Scattering	$\frac{I \text{ (obs.)}}{I_0 \text{ (Rutherford)}}$	D_1	$\frac{I \text{ (obs.)}}{I_0 \text{ (corrected)}}$
880 k.v.	20	$0.95 \pm .006$	1.002	$0.952 \pm .006$
	25	$1.08 \pm .02$	1.0025	$1.083 \pm .02$
	30	$1.25 \pm .03$	1.003	$1.254 \pm .03$
	35	$1.42 \pm .03$	1.004	$1.426 \pm .03$
	40	$1.49 \pm .04$	1.006	$1.499 \pm .04$
	45	$1.75 \pm .05$	1.009	$1.766 \pm .05$
	55	$2.44 \pm .09$	1.021	$2.499 \pm .05$
	65	$3.07 \pm .15$	1.043	$3.202 \pm .15$
	75	$4.37 \pm .35$	1.077	$4.707 \pm .35$

It may be mentioned that if as in the previous cases the idea of taking the bounded solution be given up, the correction D_2 is dropped out and so formula (10) gives for the intensity nearly same values as are obtained from Rutherford's formula (*vide* Table I). However, it should be remembered that the bounded solution was found necessary in explaining electron scattering by atoms. Now it is evident from the last column in Table I that the ratio obtained is too high to be explained by the existence of Coulomb force alone. Let us, therefore, assume tentatively that there is in addition a short range attractive force having radial symmetry as in neutron-proton interaction. Now, because the effect of exchange need not be considered unlike the case of interaction between two similar particles, we find from (5), without taking bounded solution,

$$\frac{I}{I_0} = \frac{\operatorname{cosec}^4 \tfrac{1}{2}\phi - 4\dfrac{\mu v^2}{zz'e^2}\, g \operatorname{cosec}^2 \tfrac{1}{2}\phi \cos k' r_0 + 4\left(\dfrac{\mu v^2}{zz'e^2}\, g\right)^2}{\operatorname{cosec}^4 \tfrac{1}{2}\phi} \quad \ldots \quad (13)$$

where I_0 is the intensity for Coulomb force only, ϕ is given in (6.6) and g has the same meaning as in (1) and (2). As in the present case $\mu=\frac{2}{3}m_1$ and $z=1$, $z'=2$, we have from (13), on substituting for k' and r_0 from (5·1) and (5·2),

$$\frac{I}{I_0}=\frac{\operatorname{cosec}^4\tfrac{1}{2}\phi-\frac{4}{3}\frac{m_1v^2}{e^2}g\operatorname{cosec}^2\tfrac{1}{2}\phi\cos\left\{1.35\times\frac{8\pi e^2}{hv}\left(1+\sin\frac{\phi}{2}\right)\right\}+\frac{4}{9}\left(\frac{m_1v^2}{e^2}g\right)^2}{\operatorname{cosec}^4\tfrac{1}{2}\phi} \qquad \ldots \quad (13·1)$$



Angle of Scattering
Deuteron-Helium Scattering

FIGURE 3

The theoretical values for different angles of scattering and for incident velocities 880 k.v. and 657 k.v. are obtained from (13.1), and the curves are drawn continuous in Fig. 3. The corresponding corrected experimental values (*vide* Table I, Column 5 for 880 k.v.) are given by the dotted curves. The agreement, it appears, is fairly satisfactory. The value of g used in the theoretical calculation is 2.701×10^{-13}. It is its fitting value at $75°$ for incident velocity 880 k.v.

D. *Proton-Helium Scattering*

In the case of scattering of proton by helium we have $m_2=4m_1$ and so $\mu=\frac{4}{5}m_1$ and $z=1$, $z'=2$. On substituting these values, the intensity and the critical approach are readily obtained from (7) and (7.2). The correction for the relative motion is also easily obtained, and we have

$$D_1 = \frac{16}{25} \cdot \frac{\operatorname{cosec}^4 \frac{1}{2}\theta}{\operatorname{cosec}^4 \{\frac{1}{2}\theta + \frac{1}{2}\sin^{-1}(\frac{1}{4}\sin\theta)\}} \cdot \frac{1}{\frac{1}{2}\cos\theta + \dfrac{1 + \frac{1}{16}\cos 2\theta}{\sqrt{1 - \frac{1}{16}\sin^2\theta}}} \quad \cdots \quad (14)$$

From the above we find that for $\theta = 45^\circ$, $D_1 = 1.001$. Thus the correction is negligible at least up to 45°. Consequently, Rutherford's formula (7) (after dropping the factor $\cos k'r_0$ for reasons explained before) should agree up to 45° angle of scattering. This is also supported by Heydenburg and Roberts (*l.c.*) who find that for angles of scattering between 20° and 45°, their observed intensities are very nearly equal to those given by Rutherford's formula. However, it should be noted that in using Rutherford's formula it is implicitly assumed that between proton and helium there is no short range force as in neutron-proton interaction. It is rather difficult to support this assumption in view of the fact that in all the previous cases we have already assumed the existence of such a force.

It appears to me that the above difficulties cannot be removed unless the question of bounded solution is settled. Further, we have supposed the short range force to be spherically symmetrical even for deuteron and helium. But for such complicated nuclei the force should have some axial symmetry. Attempts to improve upon the present theory on these lines will be made in future.

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